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$SO(1; n)$-twistors

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Abstract

We consider an analog of the Penrose transform for $SO(1; n)$, give for it an explicit inversion formula and connect it with the Radon transform on the sphera.

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The usual construction of twistors and the Penrose transform are connected with $SU(2; 2)$. Here we will develop twistor constructions for $SO(1; n)$ with the focus on explicit formulas. There are several multidimensional generalizations of the Penrose transform, but analogs of explicit formulas which we consider here were known only for $SU(p; q)$ [GH1] . As a consequence of these results we obtain that the Radon transform of functions on $\mathbb{R}^n$ which are extendible on the conformal compactification $\mathbb{R}C^n$ admits a holomorphic extension up to a Penrose transform. It is possible to connect another extension of the Radon transform with the projective compactification. The real version of these constructions was considered in [G2,G3].

Geometrical picture. Let $\mathbb{C}P^n_z$ be the projective space with homogeneous coordinates $z = (z_0, z_1, \dots, z_n)$ and $\mathcal{Q}_{\mathbb{C}}$ be the quadric

$$\begin{aligned}\Box(z) &= (z_0)^2 - (z_1)^2 - \cdots - (z_n)^2 \\ &= \sum_{0\le i\le n} \epsilon_i (z_i)^2 = 0, \quad \epsilon_0 = 1, \ \epsilon_i = -1, \ i > 0. \end{aligned} \tag{1}$$

Let $\mathcal{Q}_{\mathbb{R}}$ be the intersection of $\mathcal{Q}_{\mathbb{C}}$ with the real projective space $\mathbb{R}P^n$: $\Box x = 0, x \in \mathbb{R}P^n$. The quadric $\mathcal{Q}_{\mathbb{R}}$ can be identified with the sphera $S^{n-1} \subset \mathbb{R}^n$. Let

$$X = \mathcal{Q}_{\mathbb{C}} \backslash \mathcal{Q}_{\mathbb{R}}.$$

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The group $SO(1;n)$ (in the representation corresponding to the form $\Box z$) acts transitively on $\mathcal{Q}_{\mathbb{R}}$ and X. Relative to this action, X is a pseudo-Hermitian symmetric space

$$X = SO(1;n)/SO(1;n-2) \times SO(2). \tag{2}$$

We will denote $\langle\ ,\ \rangle$ the bilinear form, corresponding to the quadratic form $\Box z$. Then

$$\langle z, \epsilon\bar{z}\rangle \neq 0, \quad z \in \mathbb{C}P^n; \qquad \langle z, \bar{z}\rangle \neq 0, \quad z \in \mathcal{Q}_{\mathbb{C}}. \tag{3}$$

Dual geometric picture. We will consider the manifold M of generic hyperplane sections of $\mathcal{Q}$. Let us denote by $\mathcal{L}_\zeta$ the section of $\mathcal{Q}_{\mathbb{C}}$ by the hyperplane $\langle \zeta, z\rangle = 0$. Then M is defined in $\mathbb{C}P^n_\zeta$ by the condition $\Box\zeta \neq 0$ or we can put

$$\Box\zeta = 1. \tag{4}$$

If $\zeta = \xi + \mathrm{i}\eta$, then (4) is equivalent to

$$\Box\xi - \Box\eta = 1, \langle \xi, \eta\rangle = 0. \tag{4a}$$

The manifold M, $\dim_{\mathbb{C}} M = n$, is a pseudo-Hermitian symmetric manifold

$$M = SO(n;\mathbb{C})/SO(n-1;\mathbb{C}),$$

which is also a Stein manifold.

Let us consider the submanifold $\tilde{X} \subset M$ of such ζ that

$$\mathcal{L}_\zeta \cap \mathcal{Q}_{\mathbb{R}} = \emptyset$$

as a dual object to X. Then under condition (4) $\tilde{X}$ is defined [W] by the condition

$$\Box\xi > 0, \quad \xi_0 > 0. \tag{5}$$

To prove (5) it is convenient to use $SO(1;n)$-invariance and to consider canonical representatives of ξ. Thus $\tilde{X}$ parametrizes compact complex cycles (quadrics) of the codimension 1 inside X.

Inside $\tilde{X}$ there are several remarkable (real) homogeneous submanifolds. First, in the intersection with $\mathbb{R}^{n+1}$ we have the Riemann symmetric manifold $M_{\mathbb{R}}$ – the hyperboloid (one of two sheets):

$$\Box\xi = 1, \quad \xi_0 > 0, \quad \xi \in \mathbb{R}^{n+1}.$$

It is one of the $SO(1;n)$-orbits and we can interpret the symmetric space M as a complexification of the symmetric space $M_{\mathbb{R}}$.

The manifold M includes another real forms and between them the compact one – the sphera S^n:

$$\eta_0 = \xi_j = 0, \quad j > 0, \ (\xi_0)^2 + (\eta_1)^2 + \cdots + (\eta_n)^2 = 1.$$

The inclusion M in the projective space brings the identification of antipodal points ζ and $-\zeta$. The intersection $\tilde{X} \cap S^n$ coincides with the semisphera S^n_+ where $\xi_0 > 0$. All $SO(1;n)$-orbits in $\tilde{X}$ are intersecting S^n_+. They are parametrized by the invariant $a = \Box\xi$, $0 < a \leq 1$.

For $a = 1$ the intersection is one point and the orbit is $M_{\mathbb{R}}$, for other a the (real) dimension of the intersection is equal to $n - 1$. On the boundary of $\tilde{X}$ there is the orbit $\Omega(\tilde{X})$ – the edge of $\tilde{X}$:

$$\xi = 0, \quad \Box\eta = -1.$$

It is the pseudo-Riemannian symmetric space $SO(1; n)/SO(1; n-1)$ – the hyperboloid of one sheet. The real dimension n of $\Omega(\tilde{X})$ (and $M_{\mathbb{R}}$) coincides with the complex dimension of $\tilde{X}$. For $0 < a < 1$ the dimension of orbits is equal to $2n - 1$.

Now we will continue to investigate the boundary of $\tilde{X}$ where $\Box\xi = 0$. The manifold $\tilde{X}$ has a structure of a tube. We already had considered the edge $\Omega(\tilde{X})$. The other part of the boundary is fibering on complex components. They will be half-planes

$$\begin{aligned} \mathcal{T}(\sigma, \rho) = \{\zeta = \lambda\sigma + \mathrm{i}\rho, \Im\lambda > 0, \\ \text{where } \sigma \in \mathbb{R}^{n+1}, \ \Box\sigma = 0, \ \sigma_0 > 0, \ \rho \in \Omega(\tilde{X}), \ \langle\sigma, \rho\rangle = 0\}. \end{aligned} \tag{6}$$

Of course $\mathcal{T}$ will be conserved if we were to multiply σ by a positive constant or to add to ρ a multiple of σ. Thus $\tilde{X}$ is an $SO(1; n)$-invariant Stein manifold which is inhomogeneous.

In our analytic constructions the Stein manifold

$$\hat{X} = \{(z, \zeta); z \in X, \zeta \in \tilde{X}, \langle\zeta, z\rangle = 0\} \tag{7}$$

will play an important role. There is the natural fibering

$$\pi : \hat{X} \to X$$

whose fibers are contractible.

Let us imbed the space $\mathbb{C}P^n_\zeta$ as a quadric $\mathcal{Q}' \subset \mathbb{C}P^{n+1}_{\zeta'}$, $\zeta' = (\zeta, \zeta_{n+1})$:

$$\Box\zeta + (\zeta_{n+1})^2 = 0; \qquad \zeta \mapsto (\zeta, \mathrm{i}\sqrt{\Box\zeta}).$$

The image of M on $\mathcal{Q}'$ is defined by the condition $\zeta_{n+1} \neq 0$. On $\mathcal{Q}'$ there is a classical Cartan domain L which is biholomorphic equivalent to the Lie ball or the future tube – the connected component of

$$\langle\zeta, \bar{\zeta}\rangle + |\zeta_{n+1}|^2 > 0.$$

The image of $\tilde{X}$ will lie in L. So $\tilde{X}$ admits the natural extension up to the symmetric domain L with the group $SO(2; n)$: $\tilde{X}$ is obtained from L by removing the intersection with the hyperplane $\zeta_{n+1} = 0$. The extension $\mathcal{Q}'$ gives the compactifications of the hyperboloids $M_{\mathbb{R}}$ and $\Omega(\tilde{X})$.

The Penrose transform. We will need some invariant differential forms. Usually we will use for writing them determinants in which some columns can be 1-forms. Then for the computation of such a determinant we will take the exterior products of 1-forms. As a result the determinant of a matrix with identical columns of 1-forms may not be equal to zero. We will also denote the determinant of a matrix $(a_1, \ldots, a_n)$ through $[a_1, \ldots, a_n]$. If a column ϕ (of 1-forms) repeats r times we will write $\phi^{\{r\}}$.

In P^n we consider the Leray n-form

$$\omega_n(z) = \det(z, \mathrm{d}z, \dots, \mathrm{d}z) = [z, \mathrm{d}z^{\{n\}}] = \sum_{0 \le j \le n} (-1)^j z_j \bigwedge_{i \ne j} \mathrm{d}z_i .$$

In this matrix the column $\mathrm{d}z$ repeats n times.

On the quadric Q we will use the $(n-1)$-form

$$\omega_Q(z) = \frac{[u,\ z, \mathrm{d}z^{\{n-1\}}]}{\langle u,\ z\rangle}, \quad \Box z = 0,\ \langle u,\ z\rangle \ne 0. \tag{8}$$

The restriction of this form on Q is independent of a choice of the element u. It is possible to check directly and also it is a consequence of the fact that this form is a residue of a closed form (with a simple pole):

$$\omega_Q = c\,\mathrm{Res}_Q \frac{\omega_n(z)}{\Box z}.$$

As the element u we can choose, for example, $u = \epsilon \bar{z}$, or on X we can take $u = z$.

On the intersection of the quadric and a hyperplane $\mathcal{L}_\zeta$ we will work with the closed $(n-2)$-form

$$\omega_{Q,\zeta} = \frac{[u,\ v,\ z, \mathrm{d}z^{\{n-2\}}]}{\langle u, z\rangle\, \langle \zeta,\ v\rangle}, \quad \Box z = 0, \quad \langle \zeta,\ z\rangle = 0, \quad \langle u, z\rangle \ne 0, \quad \langle \zeta, v\rangle \ne 0. \tag{9}$$

This form on $\mathcal{L}_\zeta$ is independent of $u,\ v$ (we can take $v = \epsilon \bar{\zeta}$) and

$$\omega_{Q,\zeta} = c\,\mathrm{Res}_{\mathcal{L}_\zeta} \frac{\omega_Q(z)}{\langle \zeta,\ z\rangle}.$$

Let us define now *the generalized Penrose transform* for $(n-2)$-dimensional $\bar{\partial}$-cohomology in X with coefficients in the line bundle $\mathcal{O}(-n+2)$: $H^{(n-2)}(X, \mathcal{O}(-n+2))$. We will use the Dolbeault cohomology and consider $\bar{\partial}$-closed $(0,\ n-2)$-forms $\varphi(z,\ \bar{z};\ \mathrm{d}\bar{z})$ on X, which are homogeneous of the degree $-n+2$ in z (it means just that we take coefficients in $\mathcal{O}(-n)$). *We call the Penrose transform of* φ *the integral of* φ *on the section* $\mathcal{L}_\zeta$:

$$\hat{\varphi}(\zeta) = \int_{\mathcal{L}_\zeta} \varphi \wedge \omega_{Q,\zeta}. \tag{10}$$

The integrand is an $(n-2,\ n-2)$-form in homogeneous coordinates which can be pushed down on the cycle $\mathcal{L}_\zeta$ of the complex dimension $n-2$ in X ($(q-2)$-quadric). The result $\hat{\varphi}(\zeta)$ is a section of the line bundle $\mathcal{O}(-1)$ on $\tilde{X}$ (homogeneous of the degree -1). It is possible to verify by the direct differentiation that

$$\Box\left(\frac{\partial}{\partial \zeta}\right) \hat{\varphi}(\zeta) = 0. \tag{11}$$

The basic fact is:

Theorem. *The Penrose transform determines an isomorphism between $H^{(n-2)}(X, \mathcal{O}(-n))$ and the space of (holomorphic) sections of $\mathcal{O}(-1)$ on $\tilde{X}$ satisfying the equation*

$$\Box\left(\frac{\partial}{\partial\zeta}\right)F(\zeta)=0. \tag{11a}$$

In regards to the injectivity it is evident that $\tilde{\partial}$-exact forms transform in zero. It is nontrivial that the kernel reduces to such forms and we have only the trivial kernel in the cohomology. It is possible to prove using one of several ways which were developed for the usual Penrose transform (e.g., [GH1]) but we will not consider it here. We will focus on the proof of the surjectivity which includes some explicit formulas.

The inverse Penrose transform. As the first step of the inversion we consider the form

$$\begin{aligned}\kappa F(z,\zeta;\mathrm{d}\zeta)&=\frac{[\mathcal{D}F,\zeta,\mathrm{d}\zeta^{\{n-1\}}]}{\langle\zeta,z\rangle^{n-1}},\\ \mathcal{D}&=\epsilon z+\frac{1}{n-2}\langle z,\zeta\rangle\frac{\partial}{\partial\zeta},\quad z\in X,\ \zeta\in\tilde{X}.\end{aligned}\tag{12}$$

So κ is a differential operator of the first order out of sections of $\mathcal{O}(-1)$ on $\tilde{X}$ to differential forms of the degree $n-1$ on $\tilde{X}$ depending on $z\in X$ as parameters (all objects are holomorphic). The crucial technical fact is:

Proposition 1. *The form κF for solutions of* (11a) *is closed.*

This proposition was proved in [G2] by a direct computation. The form κF has on the submanifold $\langle z,\zeta\rangle=0$ a pole of the order $n-1$. Let us use [GH2] as a template for the computation of the residue form for multiple poles:

$$\begin{aligned}&\check{\kappa}F(z,\zeta,\mu;\mathrm{d}\zeta,\mathrm{d}\mu)\\ &\quad=\mathrm{Res}_{\langle z,\zeta\rangle=0}\kappa F(z,\zeta;\mathrm{d}\zeta)\\ &\quad=\frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial\lambda^{n-2}}\{\mathrm{d}\lambda\rfloor\lambda^{n-1}\kappa F(z,\zeta+\lambda\mu;\mathrm{d}(\zeta+\lambda\mu))\}|_{\lambda=0},\quad\langle\mu,z\rangle\neq0.\end{aligned}\tag{13}$$

The form $\check{\kappa}$ is a holomorphic form on a manifold $\check{X}$ fibering over $\hat{X}$ (7).The direct computation gives

$$\begin{aligned}&\check{\kappa}F(z,\zeta,\mu;\mathrm{d}\zeta,\mathrm{d}\mu)\\ &\quad=\frac{(n-1)!}{\langle\mu,z\rangle^{n-1}}\sum_{0\le k\le n-2}\frac{1}{k!}[\epsilon z,\zeta,\mu,\mathrm{d}\mu^{\{k\}},\mathrm{d}\zeta^{\{n-2-k\}}]\left\langle\mu,\frac{\partial}{\partial\zeta}\right\rangle^{n-2-k}F(\zeta)\\ &\qquad+\frac{(n-1)!}{(n-2)\langle\mu,z\rangle^{n-2}}\sum_{0\le k\le n-3}\frac{1}{k!(n-3-k)}\\ &\qquad\times[\partial/\partial\zeta,\zeta,\mu,\mathrm{d}\mu^{\{k\}},\mathrm{d}\zeta^{\{n-2-k\}}]\left\langle\mu,\frac{\partial}{\partial\zeta}\right\rangle^{n-3-k}F(\zeta),\\ &\qquad\langle\zeta,z\rangle=0,\ \langle\mu,z\rangle\neq0.\end{aligned}$$

Let us recall that $\mathrm{d}\mu^{\{k\}}$ means that we repeat k times the column $\mathrm{d}\mu$.This formula may seem inconvenient but it has a clear structure indeed.

The manifold $\check{X}$ is the manifold of such triples (z, ζ, μ) that

$$z \in X,\ \zeta \in \tilde{X},\ \langle \zeta, z\rangle = 0,\ \langle \mu, z\rangle \neq 0.$$

It is a Stein manifold which admits the natural fibering over X with contractible fibers

$$\rho : \check{X} \to X \tag{14}$$

and $\check{\kappa} F$ is a holomorphic closed $(n-2)$-form on $\check{X}$ with differentials along fibers of ρ only. This form is closed as the residue of a closed form. It is essential that we define the residue of holomorphic forms on $\hat{X}$ with multiple poles as some forms on an extended manifold $\check{X}$ fibering over $\hat{X}$.

Let us now take the last step of the inversion. Let Γ be a section of the fibering (14). We put

$$\tilde{\kappa} F(z, \bar{z}; \mathrm{d}\bar{z}) = \{\check{\kappa} F|_{\Gamma}\}^{(0, n-2)}. \tag{15}$$

Thus we restrict $\check{\kappa} F$ on a (real) section Γ (such sections exist since fibers of (13) are contractible), obtain an $(n-2)$-form on X and take its $(0, n-2)$-part. The result will be $\bar{\partial}$-closed form. Let us recall that the manifold $\check{X}$ is a Stein manifold but the manifold X is pseudo-concave one. Different choices of the section Γ give forms from the same cohomology class. Now we can prove the result about the inversion of the Penrose transform.

Proposition 2. *The form*

$$\varphi - c\tilde{\kappa}(\hat{\varphi}), \qquad c = \frac{(-1)^{n+1}(2\pi \mathrm{i})^{n-2}}{n-1}, \tag{16}$$

is $\bar{\partial}$-exact.

To prove this fact we will need to compute the Penrose transform of the form $\tilde{\kappa} F$ and we will show that it is equal $1/c\, F$. Then the form (16) belongs to the kernel of the Penrose transform and therefore it is $\bar{\partial}$-exact.

So we need to integrate the form $\tilde{\kappa} F$ on $\mathcal{L}_{\zeta^0}$, $\zeta^0 \in \tilde{X}$. In the computation we can select the section Γ by a most convenient way and moreover we can do it only in a small neighborhood of the $\mathcal{L}_{\zeta^0}$. So let us take the section in such a way that on $\mathcal{L}_{\zeta^0}$ we have

$$\zeta(z) \equiv \zeta^0, \qquad \mu(z) = \epsilon \bar{z}.$$

Then we need to keep in $\check{\kappa}$ only the term without differentials on ζ $(k = n-2)$. It will be

$$\frac{n-1}{|z|^{2(n-1)}} F(\zeta^0)[\epsilon z, \zeta^0, \epsilon\, \mathrm{d}\bar{z}^{\{n-2\}}], \qquad |z|^2 = \langle z, \epsilon z\rangle = |z_0|^2 + \cdots + |z_{n+1}|^2.$$

We need to multiply this form on ω_{Q,ζ^0} (we take $u = \bar{z},\ v = \epsilon\bar{\zeta}^0$) and everything reduces to the standard integral

$$\int_{\mathcal{L}_\zeta} \frac{[z,\ \epsilon\zeta, \bar{z},\ \mathrm{d}\bar{z}^{\{n-2\}}] \wedge [\bar{z},\ \epsilon\bar{\zeta}, z,\ \mathrm{d}z^{\{n-2\}}]}{|\zeta|^2\ |z|^{2(n-1)}\langle z,\ z\rangle} = \frac{1}{(2\pi\mathrm{i})^{n-2}}. \tag{17}$$

It gives us the value of the constant c. It is convenient that we pick up Γ in such a way that the integrand depends on F trivial.

Holomorphic cohomology. We will give an interpretation of our formulas on the language of holomorphic cohomology [G1,EGW1,EGW2]. Let us recall that this language gives us the ability to describe the analytic cohomology in a holomorphic language (as compared with the Čech or Dolbeault cohomology which cannot). In our example we consider on $\tilde{X}$ complex of holomorphic forms $\psi(z, \zeta, \mu; \mathrm{d}\zeta, \mathrm{d}\mu)$ with differentials along fibers of (14) and with $\mathcal{O}(-n+2)$-coefficients in z. The differentials in the complex also act along fibers (ζ, μ). We call the cohomology of this complex by *the holomorphic cohomology* of X [G1,EGW1,EGW2]: $H^{(k)}_{\mathrm{hol}}(X, \mathcal{O}(-n+2))$. It turns out that the holomorphic cohomology of X is canonically isomorphic to the Dolbeault cohomology of X. It is very general result which uses only the fact that the fibers of the fibering (14) are contractible. It is possible to write the operator from the holomorphic cohomology in the Dolbeault cohomology in a very simple and explicit form. Namely the operator on holomorphic forms of a degree k to $(0, k)$-forms:

$$\psi \mapsto \{\psi|_\Gamma\}^{(0,k)} \tag{18}$$

induces an isomorphism of cohomology. The form $\check{\kappa}F$ is just a holomorphic $(n-2)$-form of considering type and the operator (15) in $\check{\kappa}F$ is just a specialization of the operator (18).

The inverse operator out of the Dolbeault cohomology to the holomorphic cohomology can be seldom written in an explicit form. In our case we have just such an exceptional situation; the operator

$$\varphi \mapsto c\check{\kappa}(\hat{\varphi})$$

realizes such isomorphism. Thus the Penrose transform gives a possibility to construct an operator from the Dolbeault cohomology to the holomorphic one. Moreover the form $c\check{\kappa}(\hat{\varphi})$ is the canonical representative in the corresponding class of holomorphic cohomology class. Most important property of this form is that as the function of parameters z it is constant along the section $\mathcal{L}_\zeta$. Let us mention that we have here the holomorphic Hodge theorem in the sense of [EGW1].

Real forms of the operator $\Box(\partial/\partial\zeta)$. If we restrict this operator on different symmetric spaces in $\tilde{X}$, we will obtain some classical operators: the restriction on the Riemann symmetric space $M_{\mathbb{R}}$ will be the elliptic operator of Laplace–Beltrami $\Box(\partial/\partial\xi)$ and the restriction on the pseudo-Riemannian space $\Omega(\tilde{X})$ will be the wave equation $\Box(\partial/\mathrm{i}\partial\eta)$. Let us recall that we consider homogeneous functions $(\mathcal{O}(-1))$ and it corresponds to an eigen value problem.

The most important fact is that *any solution of the elliptic equation*

$$\Box\left(\frac{\partial}{\partial\xi}\right)F = 0, \quad \xi \in M_{\mathbb{R}},$$

can be extended as a holomorphic solution of

$$\Box\left(\frac{\partial}{\partial\zeta}\right)F = 0, \quad \zeta \in \tilde{X}.$$

Of course it is natural that solutions of an elliptic equation can be holomorphicaly extended in some neighborhoods but here we have the universal neighborhood where all solutions can be extended holomorphically.

The recipe of the extension is very explicit. Using F on $M_{\mathbb{R}}$ we can construct the form $\tilde{\kappa}F(z, \bar{z};\, \mathrm{d}z)$ for all $z \in X$. We need only to remark that we can choose a section Γ using only $\zeta \in M_{\mathbb{R}}$: for any $z \in X$ there is $\zeta \in M_{\mathbb{R}}$ such that $z \in \mathcal{L}_{\zeta}$. Using the homogeneity of X and $M_{\mathbb{R}}$ it is sufficient to verify for one point z. We can see also that using only the restriction F on $M_{\mathbb{R}}$ we can compute the derivatives in $\tilde{\kappa}$. The Penrose transform of $\tilde{\kappa}F$ with such a section Γ in (15) will be just the desired extension.

In fact we do not need to know F on whole $M_{\mathbb{R}}$ for the construction of a section Γ. The consideration of different sections in (15) gives a universal way to solve boundary problems: if we can reconstruct $\check{\kappa}F$ on a section Γ through boundary data we can reconstruct the solution F. We can reproduce in such a way many classical formulas for solutions of boundary problems.

If we take boundary values on the wedge $\Omega(\tilde{X})$ of solutions F on $\tilde{X}$ we obtain solutions of the wave equation $\Box(\partial/\partial\eta)F = 0$. Such boundary values there exist always in hyperfunctions.

Finally we can consider the joint boundary of $M_{\mathbb{R}}$ and $\Omega(\tilde{X})$ in $\mathcal{Q}'$ as well as the Dirichlet data on it. There is a canonical one-to-one correspondence between points of this boundary and $\mathcal{Q}_{\mathbb{R}} = S_{n-1}$ which give a possibility to interpret formula (19) below as the Poisson integral.

Conformal hyperfunctions. It is natural to interpret elements of $H^{(n-2)}(X, \mathcal{O}(-n+2))$ as hyperfunctions on $\mathcal{Q}_{\mathbb{R}} = S^{n-1}$ – *conformal hyperfunctions*. Let us construct natural imbeddings of functional spaces on $\mathcal{Q}_{\mathbb{R}}$ in the space of conformal hyperfunctions. For $f \in C^{\infty}(\mathcal{Q}_{\mathbb{R}}, \mathcal{O}(-n+2)) = C^{\infty}(S^{n-1})$ we define *the holomorphic conformal Radon transform* as

$$\hat{f}(\zeta) = \int\limits_{\mathcal{Q}_{\mathbb{R}}} f(x)\langle\zeta, x\rangle^{-1}\omega_{\mathcal{Q}}(x), \quad \zeta \in \tilde{X}. \tag{19}$$

The result is a section of $\mathcal{O}(-1)$ on $\tilde{X}$ satisfying to the wave equation (11). Therefore we can apply the operator κ (12) and its modifications and as a consequence we have the imbedding

$$C^{\infty}(\mathcal{Q}_{\mathbb{R}}, \mathcal{O}(-n+2)) \to H^{(n-2)}(X, \mathcal{O}(-n+2)) : f \mapsto \tilde{\kappa}\hat{f}. \tag{20}$$

There is a natural extension of the holomorphic conformal Radon transform on the space $\mathcal{S}'$. The crucial point is that *a distribution f is boundary values of a cohomology class* *(hyperfunction)* φ *if the Penrose transform of* φ *coincides with the holomorphic conformal Radon transform of* f:

$$\hat{\varphi} = \hat{f}.$$

We already gave the recipe of the construction of such a class for a distribution f.

For an inversion of the operator (20) let us take boundary values of $\hat{f}, \kappa\hat{f}, \check{\kappa}\hat{f}$ for $z = x \in \mathcal{Q}_{\mathbb{R}}, \zeta = \mathrm{i}\eta \in \Omega(\tilde{X})$. Such boundary values always exist in hyperfunctions but let us suppose that they exist in a classical sense. Such a way for $x \in \mathcal{Q}_{\mathbb{R}}$ we have the closed differential $(n-2)$-form

$$\check{\kappa}\hat{f}(x; \mathrm{i}\eta, \mu; \mathrm{i}\,\mathrm{d}\eta, \mathrm{d}\mu), \qquad \Box(\eta) = -1, \langle \eta, x\rangle = 0, \quad \langle \mu, x\rangle \neq 0.$$

It was proved in [G2] that for fixed $x \in \mathcal{Q}_{\mathbb{R}}$ and for any cycle γ in such manifold of (η, μ) we have

$$\int_{\gamma} \check{\kappa}\hat{f}(x; \mathrm{i}\eta, \mu; \mathrm{i}\,\mathrm{d}\eta, \mathrm{d}\mu) = c(\gamma) f(x). \tag{21}$$

If $c(\gamma) \neq 0$ then we can reconstruct f. In [G2,G3] there were considered different kind cycles γ and computed corresponding coefficients $c(\gamma)$. Using this inversion formula for holomorphic conformal Radon transform we can reconstruct boundary values of cohomology classes in distributions or functions if they exist. Let us remark that boundary values $\hat{f}(\mathrm{i}\eta), \mathrm{i}\eta \in \Omega(\tilde{X})$, will not be odd functions.

The real conformal Radon transform. We can define on $\mathcal{Q}_{\mathbb{R}} = S^{n-1}$ the real Radon transform – integrals on hyperplane sections:

$$\mathcal{R}f(\eta) = \int_{\mathcal{Q}_{\mathbb{R}}} f(x)\delta(\langle \eta, x\rangle)\omega_{\mathcal{Q}}(x). \tag{22}$$

The image of the transform will satisfy the real condition $\mathcal{O}(-1)$: $F(\rho\eta) = |\rho|^{-1}F(\eta), \rho \in \mathbb{R}\backslash 0$, and the wave equation (11). These conditions completely describe the image. There is a simple connection between the real and holomorphic conformal Radon transforms:

$$\mathcal{R}f(\eta) = \frac{1}{2\pi\mathrm{i}}(\hat{f}(\mathrm{i}\eta) + \hat{f}(-\mathrm{i}\eta)). \tag{23}$$

Concerning the inverse reconstruction, we can remark that $\hat{f}(\zeta)$ is the part of the solution $\mathcal{R}f(\eta)$ of the wave equation which is holomorphic in $\tilde{X}$. It is possible to use different integral formulas for the reconstruction of this projection. If we are interested only in boundary values $\hat{f}(\mathrm{i}\eta)$, it is convenient to remark that

$$\hat{f}(\mathrm{i}(\rho + p\sigma)) = \mathcal{R}f(\rho + p\sigma) * (p + \mathrm{i}0)^{-1},$$

where $\Box(\sigma) = 0, \langle \sigma, \rho\rangle = 0, p \in \mathbb{R}$. Using this representation we can also extend $\hat{f}$ holomorphically in half-planes $\mathcal{T}(\sigma, \rho)$ (6). It is connected with the possibility to interpret the restriction $\mathcal{R}f$ on $L_\sigma = \{\langle \eta, \sigma\rangle = 0\}$ as the affine Radon transform using the stereographic projection out of the point $\sigma \in S^{n-1}$ [G2,G3].

It is convenient for usual affine Radon transform to modify it in such a way that it will be holomorphic on one variable (using the convolution with $(p - \mathrm{i}0)^{-1}$) [G3]. It gives a possibility to write a universal inversion formula for all dimensions (even or odd). We can see that in the conformal case it is possible to modify the Radon transform in such a way that it will be the holomorphic function of all variables. The basic inversion formula uses the holomorphic version of the transform and the corresponding inversion formula is always local. Then it is possible to compute boundary values and transform the original formula in a formula for the real Radon transform which can be local or not.

Remark. We considered here cohomology with coefficients in a special line bundle $\mathcal{O}(-n+2)$. There is a standard way to consider other bundles. Let us illustrate it on the example of $\mathcal{O}(-n+1)$. So let $(0, n-2)$-form φ represent a cohomology class from $H^{(n-2)}(X, \mathcal{O}(-n+1))$. Then we define its Penrose transform as the vector-function (section of a vector bundle) $\hat{\varphi} = (\hat{\varphi}_0, \ldots, \hat{\varphi}_n)$ where

$$\hat{\varphi}_j(\zeta) = \int\limits_{\mathcal{L}_\zeta} z_j \varphi \wedge \omega_{Q,\zeta}. \tag{24}$$

We can check by the direct differentiation that $\hat{\varphi}$ satisfies to the following system of differential equations of the first order:

$$\begin{aligned} &\epsilon_k \frac{\partial \hat{\varphi}_j}{\partial \zeta_k} = \epsilon_j \frac{\partial \hat{\varphi}_k}{\partial \zeta_j}, \quad 0 \le j, k \le n, \\ &\sum_{0 \le j \le n} \epsilon_j \frac{\partial \hat{\varphi}_j}{\partial \zeta_j} = 0, \quad \epsilon_0 = 1, \ \epsilon_j = -1, \ j > 0. \end{aligned} \tag{25}$$

As the direct consequence of this system we obtain that each component of $\hat{\varphi}$ satisfies to the wave equation (11). It gives a possibility to obtain the inversion formula using the formulas which we considered above. So this system is one of the systems of first order associated with the wave equations (analogs of the massless equations). Other line bundles on X will give other associated systems of the first order.

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